Max-Share Misidentification

Supplemental Appendix

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September 30, 2025

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A Proofs of Main Results

A.1 Max-Share Setup

A.1.1 Proof of Lemma 1

Proof. The lemma follows from:

$$\delta_i' \left[\sum_{h\in\mathcal{H}} \widetilde{\Psi}_h \theta\theta' \widetilde{\Psi}_h' \right] \delta_i = \sum_{h\in\mathcal{H}} \delta_i' \widetilde{\Psi}_h \theta\theta' \widetilde{\Psi}_h' \delta_i = \sum_{h\in\mathcal{H}} \theta' \widetilde{\Psi}_h' \delta_i \delta_i' \widetilde{\Psi}_h \theta = \theta' \left[\sum_{h\in\mathcal{H}} \widetilde{\Psi}_h' \delta_i \delta_i' \widetilde{\Psi}_h \right] \theta,$$

where the second equality follows from the fact that $\delta_i' \widetilde{\Psi}_h \theta\theta' \widetilde{\Psi}_h' \delta_i$ is a product of two scalar-valued quadratic terms. □

A.1.2 Inner Product in the Frequency Domain Problem

Lemma 2. $\langle \psi_j, \psi_{j'} \rangle^{freq} \equiv \int_{\omega\in\Omega} \Gamma_{1j}^{\text{Re}}(\omega)\Gamma_{1j'}^{\text{Re}}(\omega) + \Gamma_{1j}^{\text{Im}}(\omega)\Gamma_{1j'}^{\text{Im}}(\omega) d\omega$ *is an inner product mapping* $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$.

Proof. Denoting the hth element in ψ_j by $\psi_{h,j}$ for $h \in \mathbb{N}$ and for $a, b \in \mathbb{R}$, we have:

$$\begin{aligned}
&\langle a\psi_{j_1} + b\psi_{j_2}, \psi_{j_3} \rangle^{freq} \\
&= \int_{\omega\in\Omega} \left[\sum_{h=0}^{\infty} (a\psi_{h,j_1} + b\psi_{h,j_2}) \cos(\omega h) \right] \left[\sum_{h=0}^{\infty} \psi_{h,j_3} \cos(\omega h) \right] \\
&\qquad + \left[-\sum_{h=0}^{\infty} (a\psi_{h,j_1} + b\psi_{h,j_2}) \sin(\omega h) \right] \left[-\sum_{h=0}^{\infty} \psi_{h,j_3} \sin(\omega h) \right] d\omega \\
&= a \Bigg\{ \int_{\omega\in\Omega} \left[\sum_{h=0}^{\infty} \psi_{h,j_1} \cos(\omega h) \right] \left[\sum_{h=0}^{\infty} \psi_{h,j_3} \cos(\omega h) \right] \\
&\qquad + \left[\sum_{h=0}^{\infty} \psi_{h,j_1} \sin(\omega h) \right] \left[\sum_{h=0}^{\infty} \psi_{h,j_3} \sin(\omega h) \right] d\omega \Bigg\} \\
&\quad + b \Bigg\{ \int_{\omega\in\Omega} \left[\sum_{h=0}^{\infty} \psi_{h,j_2} \cos(\omega h) \right] \left[\sum_{h=0}^{\infty} \psi_{h,j_3} \cos(\omega h) \right] \\
&\qquad + \left[\sum_{h=0}^{\infty} \psi_{h,j_2} \sin(\omega h) \right] \left[\sum_{h=0}^{\infty} \psi_{h,j_3} \sin(\omega h) \right] d\omega \Bigg\} \\
&= a\langle \psi_{j_1}, \psi_{j_3} \rangle^{freq} + b\langle \psi_{j_2}, \psi_{j_3} \rangle^{freq},
\end{aligned}$$

where the second equality follows from the linearity of the integral and summation operators and a rearrangement of terms. Thus, linearity is satisfied.

Conjugate symmetry is satisfied since:

$$\begin{aligned}\langle \psi_{j_1}, \psi_{j_2} \rangle^{freq} &= \overline{\int_{\omega \in \Omega} \Gamma^{\text{Re}}_{1j_1}(\omega)\Gamma^{\text{Re}}_{1j_2}(\omega) + \Gamma^{\text{Im}}_{1j_1}(\omega)\Gamma^{\text{Im}}_{1j_2}(\omega) d\omega} \\ &= \int_{\omega \in \Omega} \Gamma^{\text{Re}}_{1j_2}(\omega)\Gamma^{\text{Re}}_{1j_1}(\omega) + \Gamma^{\text{Im}}_{1j_2}(\omega)\Gamma^{\text{Im}}_{1j_1}(\omega) d\omega = \overline{\langle \psi_{j_2}, \psi_{j_1} \rangle^{freq}}.\end{aligned}$$

Positive-definiteness follows because for non-zero (over $\mathfrak{H}$) ψ_{j_1},

$$\langle \psi_{j_1}, \psi_{j_1} \rangle^{freq} = \int_{\omega \in \Omega} \left[\sum_{h=0}^{\infty} \psi_{h,j_1} \cos(\omega h) \right]^2 + \left[\sum_{h=0}^{\infty} \psi_{h,j_1} \sin(\omega h) \right]^2 d\omega > 0.$$

Since $\langle \cdot, \cdot \rangle^{freq}$ satisfies conjugate symmetry, linearity in the first argument, and positive-definiteness, it is an inner product operation mapping $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$. $\square$

A.1.3 Rareness of Non-unique Principal Eigenvectors

We now discuss the claim that it is rare for the max-share problem (9) to have a non-unique solution, so that the assumptions of uniqueness and a simple eigenvalue are relatively innocuous. The argument proceeds as follows.

First, note that the primitives in the max-share problem (the number of variables N, the horizon set $\mathfrak{H}$, and the frequency set Ω) could constrain the matrices of interest to be rank deficient. One example is when Ξ is rank one, in which case the solution to (9) is unique and has a closed-form expression (see Section B.1.1).

If full-rank Gram matrices are not precluded by the primitives, the space of Gram matrices with repeated eigenvalues is of measure zero within the space of all Gram matrices under the standard Lebesgue measure. This is based on two facts: (i) the space of $N \times N$ Gram matrices has a positive Lebesgue measure within the larger space of $N \times N$ real symmetric matrices and (ii) for $N \geq 2$, the space of real symmetric matrices with at least one repeated eigenvalue has a codimension of 2 in the space of real symmetric matrices and thus has a zero Lebesgue measure within that space.[1] Consequently, the intersection of these two spaces, i.e., the space of Gram matrices with at least one repeated eigenvalue, has Lebesgue measure zero and is thus rare.

[1] See, for instance, Exercise 1.3.10 in Tao (2012) for further details.

Suppose instead that our focus is limited to the set of rank-deficient Gram matrices with rank $2 \leq k \leq \bar{k}$ for some $\bar{k} \leq N-1$. Denote this set by $\mathfrak{G}_{\text{singular}}$ and consider the partition into disjoint subsets: $\mathfrak{G}_{\text{singular}} = \mathfrak{G}_{\text{rank}=N-1} \cup \mathfrak{G}_{\text{rank}=N-2} \cup \cdots \cup \mathfrak{G}_{\text{rank}=2}$. One could view $\mathfrak{G}_{\text{rank}=k}$ as a $(Nk - \frac{k(k-1)}{2})$-dimensional smooth manifold and define a positive natural surface measure on it.[2] Moreover, the additional condition of equality between two non-zero eigenvalues imposes a non-trivial algebraic constraint on the entries of the elements of $\mathfrak{G}_{\text{rank}=k}$, thus defining a proper algebraic subvariety within the manifold of $\mathfrak{G}_{\text{rank}=k}$ (see, e.g., Lax, 1998). One can then invoke a fundamental result in algebraic geometry and measure theory that a proper subvariety of a manifold has measure zero with respect to the natural surface measure of that manifold. As such, the subset of rank-k Gram matrices with repeated non-zero eigenvalues has measure zero with respect to the natural surface measure of the manifold of $\mathfrak{G}_{\text{rank}=k}$. An application of this argument to each $\mathfrak{G}_{\text{rank}=k}$ leads us to conclude that it is also rare to have Gram matrices with repeated non-zero eigenvalues within $\mathfrak{G}_{\text{singular}}$.

Together, the steps above imply that it is rare to have repeated non-zero eigenvalues in Ξ or non-unique principal eigenvectors associated with its largest (non-zero) eigenvalue, and thus rare for the max-share problem (9) to have a non-unique solution.

A.2 Proof of Theorem 1

Denote the spectrum of an arbitrary diagonalizable matrix, X, by $\text{spec}(\mathsf{X})$, the direct sum of an ordered sequence of matrices, $\{\mathsf{X}_i\}_{i \in \mathcal{I}}$, by $\bigoplus_{i \in \mathcal{I}} \mathsf{X}_i$, and the jth column of the identity matrix, I_n, by δ_j^n.

We first prove two auxiliary lemmas.

Lemma 3. *Let* $\Xi := \bigoplus_{g=1}^{G} \Xi_g$ *be a block diagonal Hermitian matrix with* G *diagonal blocks* $\{\Xi_g\}_{g=1}^{G}$*, each of size* $n_g \times n_g$*, then the following statements hold.*

(a) $\text{spec}(\Xi) = \bigcup_{g=1}^{G} \text{spec}(\Xi_g)$ *where* $\text{spec}(\Xi_g) \subset \mathbb{R}$.

(b) If v_{g_j} *is an* $n_g \times 1$ *eigenvector of the block* Ξ_g *corresponding to the eigenvalue* $\lambda_{g_j} \in \text{spec}(\Xi_g)$*, then one can construct a* $n \times 1$ *block-sparse eigenvector* v *of* Ξ *corresponding to the same eigenvalue* λ_{g_j} *by padding* $(n - n_g)$

[2]Because the property of zero determinant imposes a non-trivial constraint on a polynomial function of the entries of a real symmetric matrix, each subset $\mathfrak{G}_{\text{rank}=k}$ forms a proper algebraic subvariety in the space of all $N \times N$ real symmetric matrices and is thus of Lebesgue measure zero, rendering it inappropriate to use the Lebesgue measure to describe the rareness of Gram matrices with repeated non-zero eigenvalues within the set $\mathfrak{G}_{\text{rank}=k}$ itself.

zeros to v_g *in all block components other than those corresponding to* Ξ_g *as* $v = (\mathbf{0}'_{n_{g^-}\times 1}, v'_{g_j}, \mathbf{0}'_{n_{g^+}\times 1})'$*, where* $n_0 = n_{G+1} \equiv 0$*,* $n_{g^-} = \sum_{g'=0}^{g-1} n_{g'}$*,* $n_{g^+} = \sum_{g'=g+1}^{G+1} n_{g'}$*, and* $n = \sum_{g=1}^{G} n_g = n_{g^-} + n_{g^+} + 1$*.*

(c) *If there is a unique block* Ξ_{g_0} *such that its largest eigenvalue* $\lambda_{\max}(\Xi_{g_0})$ *is strictly larger than the largest eigenvalues of all other blocks* Ξ_g *for* $g \neq g_0$*, then the principal eigenvector* v_0 *of* Ξ *is block-sparse in the form of* $v_0 = (\mathbf{0}'_{n_{g_0^-}\times 1}, v'_{g_0}, \mathbf{0}'_{n_{g_0^+}\times 1})'$*, where* $n_0 = n_{G+1} \equiv 0$*,* $n_{g_0^-} = \sum_{g'=0}^{g_0-1} n_{g'}$*,* $n_{g_0^+} = \sum_{g'=g_0+1}^{G+1} n_{g'}$*, and* v_{g_0} *is an eigenvector of* Ξ_{g_0} *corresponding to* $\lambda_{\max}(\Xi_{g_0})$*. Additionally, if* $\lambda_{\max}(\Xi_{g_0})$ *is a simple eigenvalue in* $\operatorname{spec}(\Xi_{g_0})$ *(of algebraic multiplicity* 1*), then* v_0 *is the unique principal eigenvector of* Ξ*.*

Proof. Statement (a) follows because $\det(\Xi) = \prod_{g=1}^{G} \det(\Xi_g)$ for block diagonal matrices and the eigenvalues of a Hermitian matrix are real.

For (b), we verify $\Xi v = \lambda_{g_j} v$ as follows:

$$\Xi v = \begin{bmatrix} \left(\bigoplus_{g'=1}^{g-1} \Xi_{g'}\right) \mathbf{0}_{n_{g^-}\times 1} \\ \Xi_g v_{g_j} \\ \left(\bigoplus_{g''=g+1}^{G} \Xi_{g''}\right) \mathbf{0}_{n_{g^+}\times 1} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n_{g^-}\times 1} \\ \lambda_{g_j} v_{g_j} \\ \mathbf{0}_{n_{g^+}\times 1} \end{bmatrix} = \lambda_{g_j} \begin{bmatrix} \mathbf{0}_{n_{g^-}\times 1} \\ v_{g_j} \\ \mathbf{0}_{n_{g^+}\times 1} \end{bmatrix} = \lambda_{g_j} v.$$

For (c), it follows from (a) that $\lambda_{\max}(\Xi_{g_0}) \in \operatorname{spec}(\Xi)$ and $\lambda_{\max}(\Xi_{g_0}) = \lambda_{\max}(\Xi)$. Moreover, (b) implies that $v_0 = (\mathbf{0}'_{n_{g^-}\times 1}, v'_{g_0}, \mathbf{0}'_{n_{g^+}\times 1})'$ is an eigenvector of Ξ corresponding to $\lambda_{\max}(\Xi_{g_0})$. The uniqueness of v_0 comes from the assumption that $\lambda_{\max}(\Xi_{g_0})$ is a simple eigenvalue in $\operatorname{spec}(\Xi_{g_0})$ and thus $\operatorname{spec}(\Xi)$. □

Lemma 4. *Let* Ξ *be an* $n \times n$ *Hermitian matrix. If* δ_j^n *is an eigenvector of* Ξ *for* $1 \le j \le n$*, then* $\Xi = \Xi_{1:(j-1),1:(j-1)} \oplus \Xi_{j,j} \oplus \Xi_{(j+1):n,(j+1):n}$ *and the eigenvalue corresponding to* δ_j^n *is* $\Xi_{j,j}$*.*

Proof. First, Ξ is Hermitian, so it only has real eigenvalues. Second, by the definition of an eigenvector, the jth column, $\Xi\delta_j^n$, of Ξ must be a multiple of δ_j^n, and so is the jth row of Ξ by symmetry. As such, $\Xi = \Xi_{1:(j-1),1:(j-1)} \oplus \Xi_{j,j} \oplus \Xi_{(j+1):n,(j+1):n}$ and $\Xi\delta_j^n = \Xi_{j,j}\delta_j^n$. □

Using Lemmas 3 and 4, we can prove Theorem 1.

Proof. Sufficiency ($\Rightarrow$): If $\Xi = \Xi_{1,1} \oplus \Xi_{2:N,2:N}$ and $\Xi_{1,1} > \lambda_{\max}(\Xi_{2:N,2:N})$, then $\lambda_{\max}(\Xi) = \Xi_{1,1}$ and has multiplicity 1 in spec(Ξ). By Lemma 3 (c), the principal eigenvector of Ξ is δ_1^N and is unique.

Necessity ($\Leftarrow$): If δ_1^N is an eigenvector of Ξ, then by Lemma 4, $\Xi = \Xi_{1,1} \oplus \Xi_{2:N,2:N}$ and the eigenvalue corresponding to δ_1^N is $\Xi_{1,1}$. Now suppose that $\Xi_{1,1} < \lambda_{\max}(\Xi_{2:N,2:N})$, then by Lemma 3 (c), the principal eigenvector of Ξ cannot be δ_1^N, which leads to a contradiction. Suppose that $\Xi_{1,1} = \lambda_{\max}(\Xi_{2:N,2:N})$, then by Lemma 3 (b), one can always construct another eigenvector that corresponds to $\lambda_{\max}(\Xi) = \Xi_{1,1} = \lambda_{\max}(\Xi_{2:N,2:N})$ and is orthogonal to δ_1^N, contradicting the uniqueness of δ_1^N as the principal eigenvector of Ξ. In summary, it has to be the case that the spectral gap of Ξ is strictly positive. $\square$

A.3 Deviations from Exact Identification

A.3.1 Proof of Theorem 2

Proof. By the linearity of the inner product $\langle \cdot, \cdot \rangle$, we have, for any j,

$$\langle \psi^*, \psi_j \rangle = \sum_{k=1}^{N} \theta_k \langle \psi_k, \psi_j \rangle = \lambda_{\max}(\Xi)\theta_j, \tag{A.1}$$

where the second equality follows from the Gramian structure of $\Xi_{j,k}$ and the jth row of the eigenequation, $\Xi\theta = \lambda_{\max}(\Xi)\theta$. Summing (A.1) using weights θ_j and, again, by the linearity of the inner product, we have

$$\langle \psi^*, \psi^* \rangle = \sum_{j=1}^{N} \theta_j \langle \psi^*, \psi_j \rangle = \lambda_{\max}(\Xi) \sum_{j=1}^{N} \theta_j^2 = \lambda_{\max}(\Xi). \tag{A.2}$$

(A.1) and (A.2) jointly yield

$$\langle \psi^*, \psi_j \rangle = \theta_j \langle \psi^*, \psi^* \rangle = \theta_j \lambda_{\max}(\Xi).$$

Equation (23) follows from the linearity of the inner product.

For (24), notice that from (23) we have:

$$\left(\frac{\langle\psi^*,\hat{\psi}\rangle}{\langle\psi^*,\psi^*\rangle}\right)^2 = \left(\sum_{j=2}^{N}\alpha_j\theta_j\right)^2 \leq \left(\sum_{j=2}^{N}\alpha_j^2\right)\left(\sum_{j=2}^{N}\theta_j^2\right) = 1-\theta_1^2.$$

The inequality follows from Cauchy-Schwarz (with equality when $\alpha_j = \theta_j/\sqrt{\sum_{j=2}^{N}\theta_j^2}$). The final equality follows from $\sum_{j=2}^{N}\alpha_j^2 = \sum_{j=1}^{N}\theta_j^2 = 1$. Rearranging terms yields equation (24). □

A.3.2 Proof of Theorem 3

Proof. We first note that, under the constraint $K'\theta = 0$,

$$\theta'\Xi\theta = \theta'\check{\Xi}\theta + \theta' P_K'\Xi P_K\theta + 2\theta' P_K'\Xi M_K\theta = \theta'\check{\Xi}\theta,$$

where $P_K = I - M_K$. As such, the constrained max-share problem (25) becomes an unconstrained max-share problem

$$\arg\max_{\theta}\theta'\check{\Xi}\theta \text{ subject to } \theta'\theta = 1. \tag{A.3}$$

It is easy to see that $\check{\Xi} = M_K\Xi M_K$ remains Hermitian. Moreover, by direct calculation, the (j,j') element of $\check{\Xi}$ is $\check{\Xi}_{jj'} = \langle\sum_{k=1}^{N} M_{K,kj}\psi_k, \sum_{k=1}^{N} M_{K,kj'}\psi_k\rangle$, which inherits the same inner product operation from the definition of Ξ as in Theorem 1 but over a different set of vectors $\{\check{\psi}_j\}_{j=1}^{N}$, where $\check{\psi}_j \equiv \sum_{k=1}^{N} M_{K,kj}\psi_k$. Theorem 1 thus applies in (A.3), as long as δ_1^N is attainable.

Sufficiency ($\Rightarrow$): If $K'\delta_1^N = 0$, δ_1^N is attainable, the sufficiency part of Theorem 1 (with Ξ replaced by $\check{\Xi}$) implies that δ_1^N is the unique solution to (25).

Necessity ($\Leftarrow$): If δ_1^N is the unique solution to (25), it has to satisfy the constraint in (25), so $K'\delta_1^N = 0$. The necessity part of Theorem 1 (with Ξ replaced by $\check{\Xi}$) implies that $\check{\Xi} = \check{\Xi}_{1,1}\oplus\check{\Xi}_{2:N,2:N}$ and $\check{\Xi}_{1,1} > \lambda_{\max}(\check{\Xi}_{2:N,2:N})$. □

A.3.3 Proof of Corollary 1

Proof. In the constrained max-share problem (25), suppose that the feasibility condition $K'\delta_1 = 0$ holds, then M_K is block-diagonal:

$$M_K = 1 \oplus M_{K_{2:N,:}},$$

where $K_{2:N,:}$ selects all but the first row of K and $M_{K_{2:N,:}}$ is the annihilator matrix for $K_{2:N,:}$, that is, $M_{K_{2:N,:}} = \left(I_{N-1} - K_{2:N,:}\left(K'_{2:N,:}K_{2:N,:}\right)^{-1}K'_{2:N,:}\right)$. Therefore, $\sum_{k=1}^{N} M_{K,k1}\psi_k = \psi_1$ and $\check{\Xi}_{1,1} = \Xi_{1,1}$.

The orthogonality condition in Theorem 3 becomes

$$\langle\psi_1, \sum_{k=2}^{N} M_{K,kj'}\psi_k\rangle = 0 \text{ for all } j' = 2, 3, \ldots, N, \tag{A.4}$$

which is implied by the orthogonality condition in Theorem 1 and the linearity of the inner product operation. On the other hand, without the orthogonality condition in Theorem 1, for any $l = 1, 2, \ldots, m$, the lth constraint (column l of K) implies that

$$\sum_{j'=2}^{N} K_{j',l}\langle\psi_1, \sum_{k=2}^{N} M_{K,kj'}\psi_k\rangle = 0.$$

Therefore, the equations (A.4) are linearly dependent with m degrees of slackness. Thus, the orthogonality condition in Theorem 1 is sufficient but not necessary for the orthogonality condition in Theorem 3.

Furthermore, note that $\check{\Xi}_{2:N,2:N}$ can be expressed as $M_{K_{2:N,:}}\Xi_{2:N,2:N}M_{K_{2:N,:}}$, and it is easy to see that $M_{K_{2:N,:}}$ is an orthogonal projection matrix. By a variant of the Poincaré Separation Theorem,[3] we have

$$\lambda_{\max}(\check{\Xi}_{2:N,2:N}) \leq \lambda_{\max}(\Xi_{2:N,2:N}).$$

Given $\check{\Xi}_{1,1} = \Xi_{1,1}$, it is immediate to see that the relative size condition in Theorem 1 is sufficient but not necessary for the relative size condition in Theorem 3. □

[3]This is also known as the Cauchy Interlacing Theorem; see Theorem 11.11 in Magnus and Neudecker (2019).

A.3.4 Proof of Proposition 1

Proposition 1 is a special case of a standard result for the differential of eigenvalues and eigenvectors.[4] We reproduce it in a form that facilitates the interpretation of our results. We use $\|\cdot\|$ to denote the Euclidean (Frobenius) norm for vectors (matrices).

Lemma 5. *Let Ξ_0 be a real symmetric $n \times n$ matrix with a complete set of orthornormal eigenvectors $\{v_{0j}\}_{j=1}^n$ that span $\mathbb{R}^n$. Suppose v_{0i} is the eigenvector corresponding to a simple eigenvalue λ_{0i} of Ξ_0 for some $1 \leq i \leq n$. When Ξ_0 is perturbed by an infinitesimal and symmetric $\mathrm{d}\Xi = o(\|\Xi_0\|)$, the following first order perturbation result holds for the new eigenpair (λ_i and v_i such that $\Xi v_i = \lambda_i v_i$) of the perturbed matrix, $\Xi := \Xi_0 + \mathrm{d}\Xi$:*

$$\lambda_i = \lambda_{0i} + v_{0i}' \mathrm{d}\Xi v_{0i} + O\left(\|\mathrm{d}\Xi\|^2\right),$$
$$v_i = v_{0i} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{v_{0j}'\mathrm{d}\Xi v_{0i}}{\lambda_{0i} - \lambda_{0j}} v_{0j} + O\left(\|\mathrm{d}\Xi\|^2\right).$$

Proof. By Theorem 8.9 in Magnus and Neudecker (2019), the functions $\lambda(\Xi)$ and $v(\Xi)$ are infinitely differentiable in the neighborhood $N(\Xi_0) \subset \mathbb{R}^{n\times n}$ of Ξ_0. As such, it is valid to neglect higher order terms and focus on the first order approximation to the exact variation of λ_{0i} and v_{0i} in the event of an infinitesimal perturbation $\mathrm{d}\Xi$ to Ξ_0, that is, $\mathrm{d}\lambda_i$ and $\mathrm{d}v_i$ in $\lambda_i = \lambda_{0i} + \mathrm{d}\lambda_i + O\left(\|\mathrm{d}\Xi\|^2\right)$, and $v_i = v_{0i} + \mathrm{d}v_i + O\left(\|\mathrm{d}\Xi\|^2\right)$.

By construction, the perturbed matrix Ξ is symmetric and it is thus without loss of generality possible to normalize its eigenvectors $\{v_j\}_{j=1}^n$ such that

$$v_i'v_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}. \tag{A.5}$$

Neglecting higher order terms, (A.5) implies

$$\mathrm{d}v_i' v_{0j} + v_{0i}' \mathrm{d}v_j = 0, \text{ for } 1 \leq i, j \leq n. \tag{A.6}$$

By the definition of eigenvalue and eigenvector of Ξ, $\Xi v_i = \lambda_i v_i$. Substituting in

[4]See, for instance, Magnus and Neudecker (2019) for a textbook exposition for symmetric matrices, Aït-Sahalia and Xiu (2019) for its application in PCA analysis of high-frequency data, and Greenbaum, Li, and Overton (2020) for a comprehensive treatment for general square matrices.

$\Xi = \Xi_0 + \mathrm{d}\Xi$ and the expansions of λ_i and v_i, and neglecting higher order terms, we obtain

$$\Xi_0 \mathrm{d}v_i + \mathrm{d}\Xi v_{0i} = \lambda_{0i}\mathrm{d}v_i + \mathrm{d}\lambda_i v_{0i}. \tag{A.7}$$

Left multiplying (A.7) with v_{0i}', we have

$$v_{0i}'\Xi_0 \mathrm{d}v_i + v_{0i}'\mathrm{d}\Xi v_{0i} = v_{0i}'\lambda_{0i}\mathrm{d}v_i + \mathrm{d}\lambda_i v_{0i}'v_{0i},$$

which, together with the facts that $\Xi_0 v_{0i} = \lambda_{0i}v_{0i}$ and $v_{0i}'v_{0i} = 1$, yields $\mathrm{d}\lambda_i = v_{0i}'\mathrm{d}\Xi v_{0i}$.

Now, note that $\{v_{0j}\}_{j=1}^n$ is orthonormal and spans $\mathbb{R}^n$, and $\mathrm{d}v_i \in \mathbb{R}^n$, so $\mathrm{d}v_i = \sum_{j=1}^n c_{ij}v_{0j}$, where $c_{ii} = v_{0i}'\mathrm{d}v_i = 0$ from (A.6) taking $j = i$, and for $j \neq i$, c_{ij} is determined as follows. Pre-multiplying (A.7) with v_{0j}', we have

$$v_{0j}'\Xi_0 \mathrm{d}v_i + v_{0j}'\mathrm{d}\Xi v_{0i} = v_{0j}'\lambda_{0i}\mathrm{d}v_i + \mathrm{d}\lambda_i v_{0j}'v_{0i},$$

which, together with the facts that $\Xi_0 v_{0j} = \lambda_{0j}v_{0j}$ and $v_{0j}'v_{0i} = 0$, yields

$$c_{ij} = v_{0j}'\mathrm{d}v_i = \frac{v_{0j}'\mathrm{d}\Xi v_{0i}}{\lambda_{0i} - \lambda_{0j}}.$$

Therefore, we have obtained

$$\mathrm{d}v_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{v_{0j}'\mathrm{d}\Xi v_{0i}}{\lambda_{0i} - \lambda_{0j}} v_{0j}.$$

□

We now specialize the results of Lemma 5 to our setting for Proposition 1.

Proof. $\Xi_0 \equiv \Xi_{1,1} \oplus \Xi_{2:N,2:N}$ is a real symmetric $N \times N$ matrix with the largest eigenvalue $\Xi_{1,1}$ and the principal eigenvector δ_1^N from Theorem 1. The perturbation $\mathrm{d}\Xi = o\left(\|\Xi_0\|\right)$ by assumption. Thus, it is valid to specialize Lemma 5 to $\Xi = \Xi_0 + \mathrm{d}\Xi$ by treating Ξ_0 as the unperturbed matrix and Ξ as the perturbed matrix.

By Lemma 3 (a), $\{\lambda_{0j}\}_{j=2}^N$ are also eigenvalues of Ξ_0. Also, define $v_{0j} = (0, w_{0j}')'$ for $2 \leq j \leq N$. Then, $\delta_1^N \perp v_{0j}$ for any j and thus $\{\delta_1^N, \{v_{0j}\}_{j=2}^N\}$ constitutes a complete set of orthonormal eigenvectors of Ξ_0.

Denote the principal eigenvector of Ξ by v_1, which is unique by the assumption that $\lambda_{\max}(\Xi)$ is simple. By Lemma 5 and up to a first order approximation, $\lambda_{\max}(\Xi) - \lambda_{\max}(\Xi_0) = \left(\delta_1^N\right)' \mathrm{d}\Xi \delta_1^N + O\left(\|\mathrm{d}\Xi\|^2\right) = 0 + O\left(\|\mathrm{d}\Xi\|^2\right) = O\left(\|\nu\|^2\right)$ by construction. Thus,

$$
\begin{aligned}
v_1 &= \delta_1^N + \sum_{j=2}^{N} \frac{v_{0j}' \mathrm{d}\Xi \delta_1^N}{\Xi_{1,1} - \lambda_{0j}} v_{0j} + O\left(\|\nu\|^2\right) \\
&= \delta_1^N + \sum_{j=2}^{N} \frac{w_{0j}' \nu}{\Xi_{1,1} - \lambda_{0j}} v_{0j} + O\left(\|\nu\|^2\right)
\end{aligned}
$$

□

A.3.5 Proof of Proposition 2

Proposition 2 is an application of the Davis-Kahan $\sin\theta$ theorem (Davis and Kahan, 1970), which yields bounds on the distance between subspaces spanned by population eigenvectors and their sample analogs. These correspond, respectively, to eigenvectors of the unperturbed and perturbed matrices in our context. However, their bounds are usually in terms of eigengaps between certain population and sample eigenvalues. One of its variants (Yu et al., 2015) is then often used to derive such bounds in terms of the population eigenvalue separation condition and a smaller matrix norm (minimum of a scaled operator norm and the Frobenius norm). In the following auxiliary lemma, we specialize these bounds to study the change of the principal eigenvector in our context. We denote the operator norm for matrices by $\|\cdot\|_{\mathrm{op}}$, i.e., $\|\mathsf{X}\|_{\mathrm{op}} \equiv \inf\{c > 0 : \|\mathsf{X}v\| \leq c\,\|v\| \text{ for all } v \in \mathcal{V}\}$ for the normed vector space $\mathcal{V}$.

Lemma 6. *Let Ξ_0 be a real symmetric $n \times n$ matrix, with eigenvalues $\lambda_{01} > \lambda_{02} \geq \cdots \geq \lambda_{0n}$. Suppose Ξ_0 is perturbed by a symmetric $\mathrm{d}\Xi$, resulting in another symmetric matrix Ξ, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. If v_{01} and v_1 are the (normalized) principal eigenvector of Ξ_0 and Ξ, respectively and denote the principal angle between them by $\Theta(v_{01}, v_1)$, then*

$$
\sin\Theta(v_{01}, v_1) \leq \frac{2\,\|\mathrm{d}\Xi\|_{\mathrm{op}}}{\lambda_{01} - \lambda_{02}} = \frac{2\lambda_{\max}(\mathrm{d}\Xi)}{\lambda_{01} - \lambda_{02}}.
$$

Moreover, if $v_1' v_{01} \geq 0$*, then*

$$\|v_1 - v_{01}\| \leq \frac{2^{3/2}\,\|\mathrm{d}\Xi\|_{\mathrm{op}}}{\lambda_{01} - \lambda_{02}} = \frac{2^{3/2}\lambda_{\max}(\mathrm{d}\Xi)}{\lambda_{01} - \lambda_{02}}.$$

Proof. This is a direct implication of Corollary 1 in Yu et al. (2015) for $j = 1$. The bound on the Euclidean distance between the principal eigenvectors is by the fact that

$$\begin{aligned}\|v_1 - v_{01}\|^2 &= 2 - 2v_1' v_{01} = 2\left[1 - \cos\Theta(v_{01}, v_1)\right] \\ &\leq 2\left[1 - \cos^2\Theta(v_{01}, v_1)\right] = 2\sin^2\Theta(v_{01}, v_1).\end{aligned}$$

□

Using the above Lemma 6, we now prove Proposition 2.

Proof. This is a direct application of Lemma 6 and Theorem 1 (which gives rise to the principal eigenvector and the eigengap for the unperturbed matrix $\Xi_0 \equiv \Xi_{1,1} \oplus \Xi_{2:N,2:N}$). It remains to calculate $\|\mathrm{d}\Xi\|_{\mathrm{op}}$. By the formula for the determinant of a block matrix, one has $\det(\lambda)\det\left(\lambda I_{N-1} - \frac{1}{\lambda}\nu\nu'\right) = 0$, whose largest nonzero root is $\|\nu\|$ and thus $\|\mathrm{d}\Xi\|_{\mathrm{op}} = \|\nu\|$. □

B Supplementary Results

B.1 Special Cases

We now characterize the solution to the max-share problem, (9) in two special cases.

B.1.1 Rank One

Ξ is rank one when the impulse responses of the target variable to all shocks have same shape. A particular case is in the time domain problem with only one horizon in the set $\mathcal{H}$. It represents the most severe violation of orthogonality with weights characterized by the following lemma.

Lemma 7. *Suppose* Ξ *is rank one. Then the solution to the max-share problem,* (9)*, satisfies:*

$$\theta_j \propto \sqrt{\Xi_{j,j}}. \tag{B.1}$$

Proof. Write $\Xi = vv'$. The unique non-zero eigenvalue is $\sum_j v_j^2$. By the definition of the eigenvector, we have $\Xi\theta = \sum_j v_j^2 \theta$, and thus $v_j \sum_{j'} v_{j'} \theta_{j'} = \theta_j \sum_{j'} v_{j'}^2$, or

$$\frac{\theta_j}{\sqrt{\Xi_{j,j}}} = \frac{\theta_j}{v_j} = \frac{\sum_{j'} v_{j'} \theta_{j'}}{\sum_{j'} v_{j'}^2}$$

for all j. □

Lemma 7 states that when Ξ is rank one, the contamination to the identified shock is proportional to the size of impulse responses to each of the shocks. Although inconsequential for the identified response of the target variable to the max-share shock, the convolution of shocks can substantially impact other quantities including the impulse responses of other variables and forecast error or dynamic variance decompositions. The rank one case illustrates a more general result that the principal eigenvector of Ξ will load on combinations of shocks with similarly shaped responses, emphasizing the role of the orthogonality condition in Theorem 1.

B.1.2 Two Shocks

With $N = 2$, we have an analytic solution for the max-share problem, (9), even when the identification conditions are not satisfied. The formulas provide intuition for how violations to orthogonality and relative size matter.

Lemma 8. *Suppose $N = 2$ and assume without loss of generality that $\Xi_{1,2} > 0$. Then the solution to the max-share problem,* (9)*, implies:*

$$\frac{\theta_1}{\theta_2} = \frac{\vartheta + \sqrt{\vartheta^2 + 4}}{2} \quad \textit{where} \quad \vartheta \equiv \frac{\Xi_{1,1} - \Xi_{2,2}}{\Xi_{1,2}}. \tag{B.2}$$

Proof. The eigenvector θ satisfies $(\Xi - \lambda I)\theta = 0$, which implies:

$$\lambda = \Xi_{1,1} + \Xi_{1,2}\frac{\theta_2}{\theta_1} = \Xi_{2,2} + \Xi_{1,2}\frac{\theta_1}{\theta_2}.$$

Multiplying throughout by θ_1/θ_2 and dividing by $\Xi_{1,2}$, we have:

$$\left(\frac{\theta_1}{\theta_2}\right)^2 - \vartheta\left(\frac{\theta_1}{\theta_2}\right) - 1 = 0,$$

where $\vartheta \equiv \frac{\Xi_{1,1}-\Xi_{2,2}}{\Xi_{1,2}}$. The quadratic equation has two solutions, with (B.2) corresponding to the larger eigenvalue. $\square$

The expression for θ_1/θ_2 is an increasing function of ϑ. The role of the orthogonality condition is captured by $\Xi_{1,2}$ in the denominator of ϑ. In particular, with $\Xi_{1,2} = 0$, the ratio θ_1/θ_2 tends to either zero or infinity. In contrast, $\Xi_{1,2} = \sqrt{\Xi_{1,1}\Xi_{2,2}}$ corresponds to the rank one case. The numerator of ϑ captures the relative size condition. Substantial weight will be placed on Shock 1 if $\Xi_{1,1} - \Xi_{2,2}$ is sufficiently large, i.e., the response to Shock 1 is sufficiently large relative to Shock 2 at the chosen horizons or frequencies.

When $\Xi_{1,2} = 0$, then the eigenvectors and associated eigenvalues are δ_j and $\Xi_{j,j}$ for $j \in \{1, 2\}$. Orthogonality is then satisfied and the relative size condition requires that $\Xi_{1,1} > \Xi_{2,2}$.

B.2 Extension of Theorem 2 to Constrained Problem

The following theorem extends Theorem 2 to the constrained problem. As before, denote the annihilator matrix by $M_K \equiv I - K(K'K)^{-1}K'$, and define $\check{\Xi} = M_K \Xi M_K$ and $\check{\psi}_j = \sum_{k-1}^{N} M_{K,kj}\psi_k$ for $j = 1, \ldots, N$.

Theorem 4. *Suppose the constrained max-share problem* (25) *has a unique solution* $\theta = (\theta_1, \ldots, \theta_N)'$ *with associated largest eigenvalue* $\lambda_{\max}(\check{\Xi})$ *and max-share impulse response* $\check{\psi}^* = \sum_{k=1}^{N} \theta_k \check{\psi}_k$. *Then for an impulse response* $\hat{\psi} \equiv \sum_{j=2}^{N} \alpha_j \psi_j$ *with* $\sum_{j=2}^{N} \alpha_j^2 = 1$, *we have:*

$$\check{A} M_K \check{P} = \lambda_{\max}\left(\check{\Xi}\right) \check{W} \tag{B.3}$$

and the following upper bound for the weight on the targeted shock:

$$\theta_1^2 \leq 1 - \lambda_{\max}\left(\check{\Xi}\right)^{-2} \left\|\check{A} M_K \check{P}\right\|^2, \tag{B.4}$$

where we have defined:

$$\check{A} \equiv \begin{bmatrix} 0 & \mathbf{0}_{1\times(N-1)} \\ \mathbf{0}_{(N-1)\times 1} & \operatorname{diag}(\alpha_2, \ldots, \alpha_N) \end{bmatrix}, \quad \check{P} = \begin{bmatrix} \langle \check{\psi}^*, \psi_1 \rangle \\ \vdots \\ \langle \check{\psi}^*, \psi_N \rangle \end{bmatrix}, \quad \check{W} = \begin{bmatrix} 0 \\ \alpha_2\theta_2 \\ \vdots \\ \alpha_N\theta_N \end{bmatrix}.$$

Proof. As in the proof of Theorem 3, $\breve{\Xi}$ remains Gramian. We thus have, for any j,

$$\langle \breve{\psi}^*, \breve{\psi}_j \rangle = \sum_{k=1}^{N} \theta_k \langle \breve{\psi}_k, \breve{\psi}_j \rangle = \lambda_{\max}(\breve{\Xi})\theta_j, \tag{B.5}$$

where the second equality follows from the Gramian structure of $\breve{\Xi}_{j,k}$ and the jth row of the eigenequation ($\breve{\Xi}\theta = \lambda_{\max}(\breve{\Xi})\theta$). Now, summing up (B.5) using weights θ_j and, again, by the linearity of the inner product, we have:

$$\langle \breve{\psi}^*, \breve{\psi}^* \rangle = \sum_{j=1}^{N} \theta_j \langle \breve{\psi}^*, \breve{\psi}_j \rangle = \lambda_{\max}(\breve{\Xi}) \sum_{j=1}^{N} \theta_j^2 = \lambda_{\max}(\breve{\Xi}). \tag{B.6}$$

(B.5), (B.6), and the definition of $\breve{\psi}_j$ jointly yield:

$$\langle \breve{\psi}^*, \breve{\psi}_j \rangle = \theta_j \langle \breve{\psi}^*, \breve{\psi}^* \rangle = \theta_j \lambda_{\max}(\breve{\Xi}) = \sum_{k=1}^{N} M_{K,kj} \langle \breve{\psi}^*, \psi_k \rangle, \tag{B.7}$$

where $j = 1, 2, \ldots, N$. The matrix form of (B.7) implies:

$$\breve{A} M_K \breve{P} = \lambda_{\max}(\breve{\Xi}) \breve{A} \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_N \end{bmatrix} = \lambda_{\max}(\breve{\Xi}) \breve{W}, \tag{B.8}$$

which is exactly (B.3). Taking the norm of both sides of (B.8) and applying the Cauchy-Schwarz inequality, we have:

$$\begin{aligned} \lambda_{\max}\left(\breve{\Xi}\right)^{-2} \left\| \breve{A} M_K \breve{P} \right\|^2 = \left\| \breve{W} \right\|^2 = \sum_{j=2}^{N} \alpha_j^2 \theta_j^2 \\ \leq \left(\sum_{j=2}^{N} \alpha_j^2 \right) \left(\sum_{j=2}^{N} \theta_j^2 \right) = 1 - \theta_1^2, \end{aligned}$$

which yields (B.4). □

B.3 Supply and Demand

Model Impulse Responses. Normalizing the coefficient on the supply shock innovations, we can write the supply and demand system (37)-(38) as a VAR(1):

$$Y_t = GFG^{-1}Y_{t-1} + \begin{bmatrix} \gamma^d & \gamma^s \\ -1 & 1 \end{bmatrix} Q\varepsilon_t, \tag{B.9}$$

where

$$Y_t = \begin{bmatrix} q_t \\ p_t \end{bmatrix}, \; F = \begin{bmatrix} \rho^s & 0 \\ 0 & \rho^d \end{bmatrix}, \; Q = \begin{bmatrix} \sigma^s & 0 \\ 0 & \sigma^d \end{bmatrix}, \text{ and } G = \frac{1}{\gamma^s + \gamma^d} \begin{bmatrix} \gamma^d & \gamma^s \\ -1 & 1 \end{bmatrix}.$$

The innovations ε_t are iid standard normal and ρ^d is the persistence of the demand shock. The true impulse response at horizon h of the ith variable to a one unit innovation in the jth shock is the (i, j) element of:

$$\Psi_h = GF^hQ = \frac{1}{\gamma^s + \gamma^d} \begin{bmatrix} (\rho^s)^h\gamma^d\sigma^s & (\rho^d)^h\gamma^s\sigma^d \\ -(\rho^s)^h\sigma^s & (\rho^d)^h\sigma^d \end{bmatrix}. \tag{B.10}$$

Each of the elements of Ψ_h has the form $(\rho^{\mathsf{x}})^h\mathsf{q}$ where $\mathsf{x} \in \{s, d\}$ and q is the on-impact response, as in an AR(1).

Frequency Domain Results. In the frequency domain, we follow Angeletos et al. (2020) and target the response of output at frequency band $\Omega = \left[\frac{2\pi}{32}, \frac{2\pi}{6}\right]$. They label this as the "main business cycle" shock, finding the striking result that the identified shock produces a large responses in real variables but a small response in inflation.

First, we consider the case with symmetric processes for the supply and demand shocks:

$$\gamma^s = \gamma^d = 1.00, \qquad \rho^s = \rho^d = 0.95, \qquad \sigma^s = \sigma^d = 1.00.$$

The top panel of Figure B.1 shows that the max-share shock qualitatively resembles the main business cycle shock in Angeletos et al. (2020), producing a positive response in output, q_t, but no response in price, p_t. The residual shock displays the opposite behavior, producing a zero response in output but a substantial response in price. In this model, the lack of response in output to the residual shock occurs as long as the

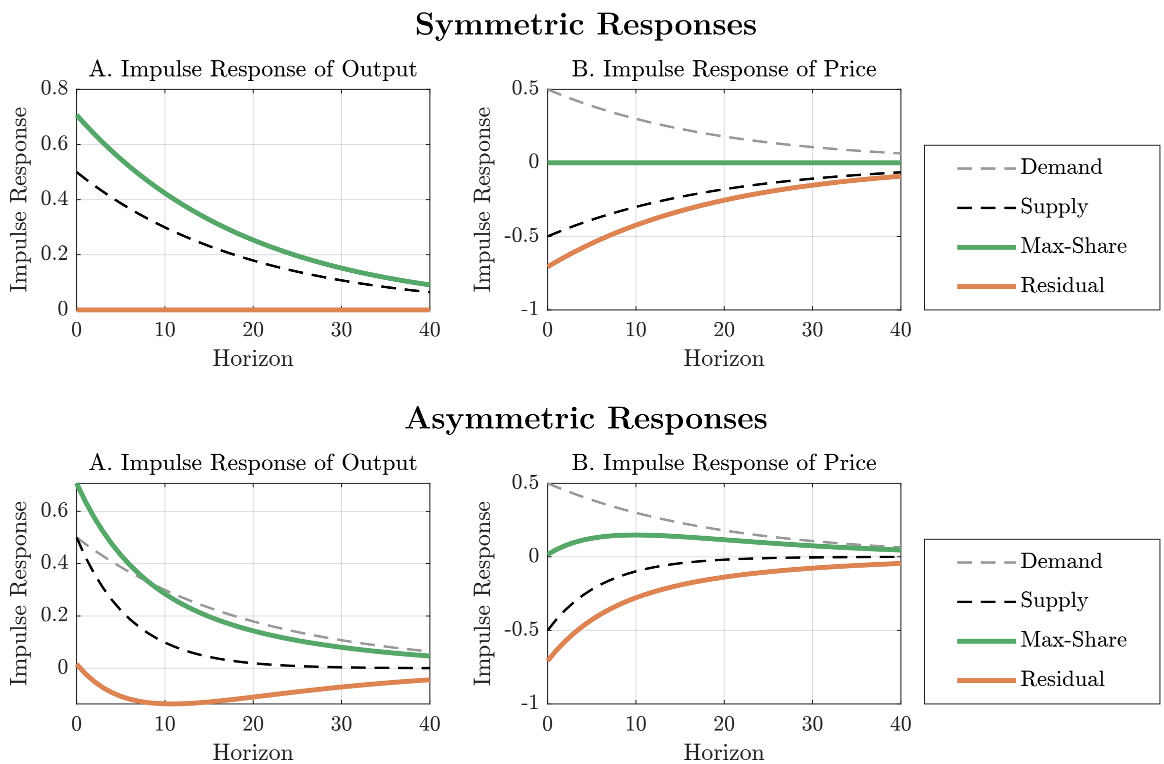

Figure B.1: Main business cycle shock in supply and demand example identified via max-share in the frequency domain. **Top panels:** Symmetric responses with $\rho^s = \rho^d = 0.95$; **Bottom panels:** Asymmetric responses with $\rho^s = 0.85$ and $\rho^d = 0.95$. Dashed lines indicate true responses; green and orange solid lines correspond to identified max-share and residual shocks, respectively.

two true underlying shocks have the same persistence, $\rho^s = \rho^d$. Since this implies that the response of output to both the max-share and residual shocks must have AR(1) dynamics with persistence $\rho^s = \rho^d$, the residual shock is forced to produce a zero response. The responses of price then depend on the elasticities, $\{\gamma^s, \gamma^d\}$, and standard deviations, $\{\sigma^s, \sigma^d\}$. In this context, valid max-share identification rests on the argument that the one other shock in the economy affects prices but not quantities.

We deviate from this knife-edge case by setting $\rho^s \neq \rho^d$. The lower panel of Figure B.1 shows results when we choose $\rho^s = 0.85$ but keep all other parameters unchanged. The max-share shock now produces a positive but relatively small response in price. However, the residual shock also generates impulse responses in output and price that have the same sign. In other words, with this parameterization, max-share identification implies that both the supposed main business cycle shock of Angeletos et al. (2020) and the residual shock have the features of demand shocks, essentially ruling out the presence of supply shocks.

This message echoes our findings from the time domain—in using max-share identification, researchers need to be careful of implications for not only the targeted shock, but also the untargeted ones. In both cases discussed here, the orthogonality conditions lead to untargeted shocks with responses that seem unlikely from economic theory.

B.4 Empirical Application

Figures B.2 and B.3 show the contributions of the shocks to the FEV of each variable, providing further suggestive evidence that the shocks are not all be cleanly identified.

First, the total contribution of the shocks to FEVs sums to more than one for certain variables and horizons at the posterior median. This occurs in the time domain for TFP at around horizon 20 and for GDP at around horizon 12. In the frequency domain, we similarly see this at certain business cycle frequencies for TFP, consumption, and GDP.

Second, while the max-share shock does account for a substantial fraction of the FEV at the targeted horizons and frequencies, there are also other shocks that individually have nontrivial FEV contributions. For example, the TFP surprise shock accounts for around 1/3 to the FEV of TFP at horizon 40, the target horizon for the TFP news shock. In addition, the TFP news shock accounts for around 1/3 at frequency $2\pi/32$. These contributions suggest that if orthogonality is not satisfied, max-share could place a sizable weight on these shocks.

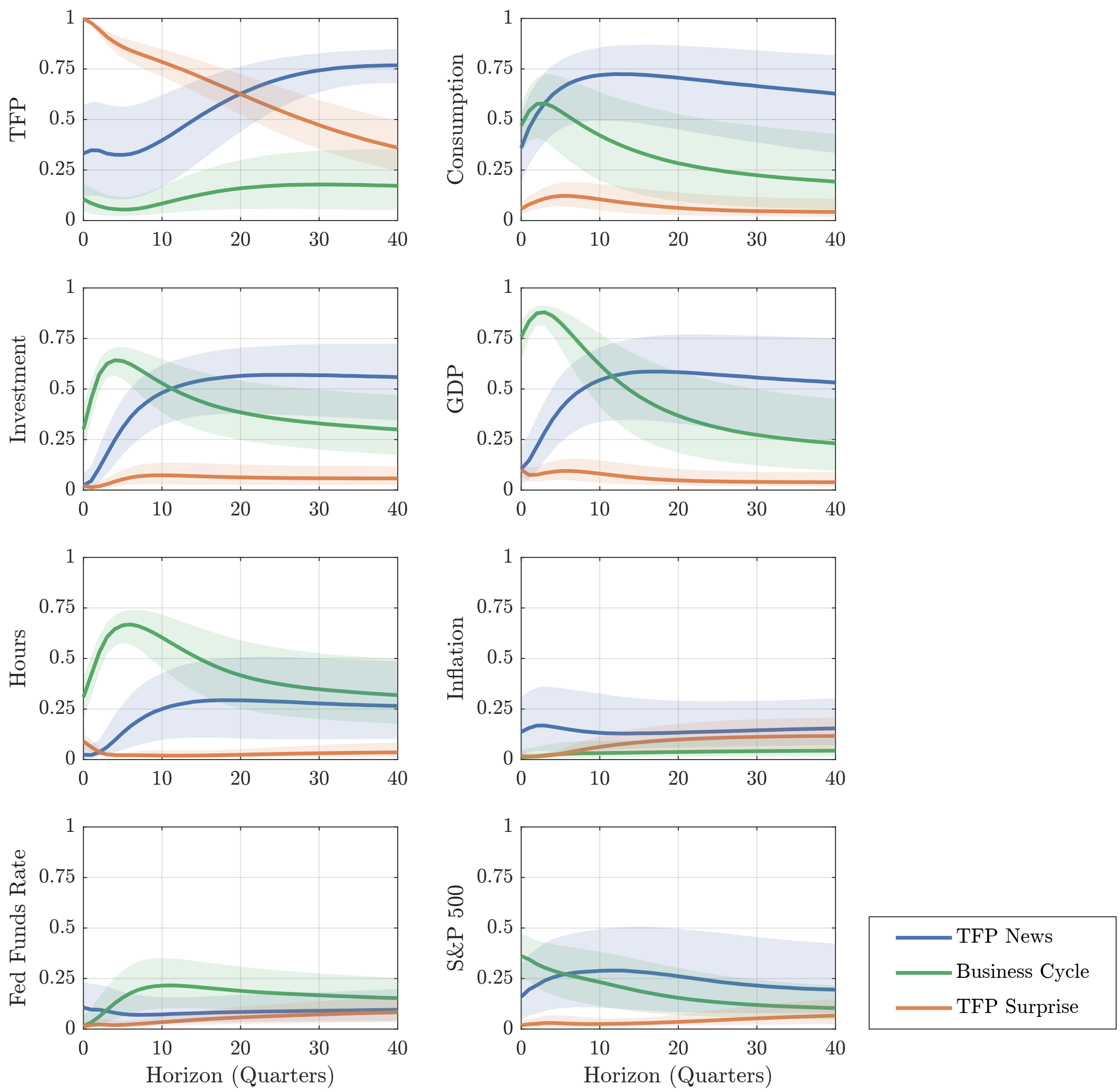

Figure B.2: Posterior estimates for FEV contributions of identified shocks in the time domain. **Solid lines:** Median response; **Shaded regions:** 68% error bands.

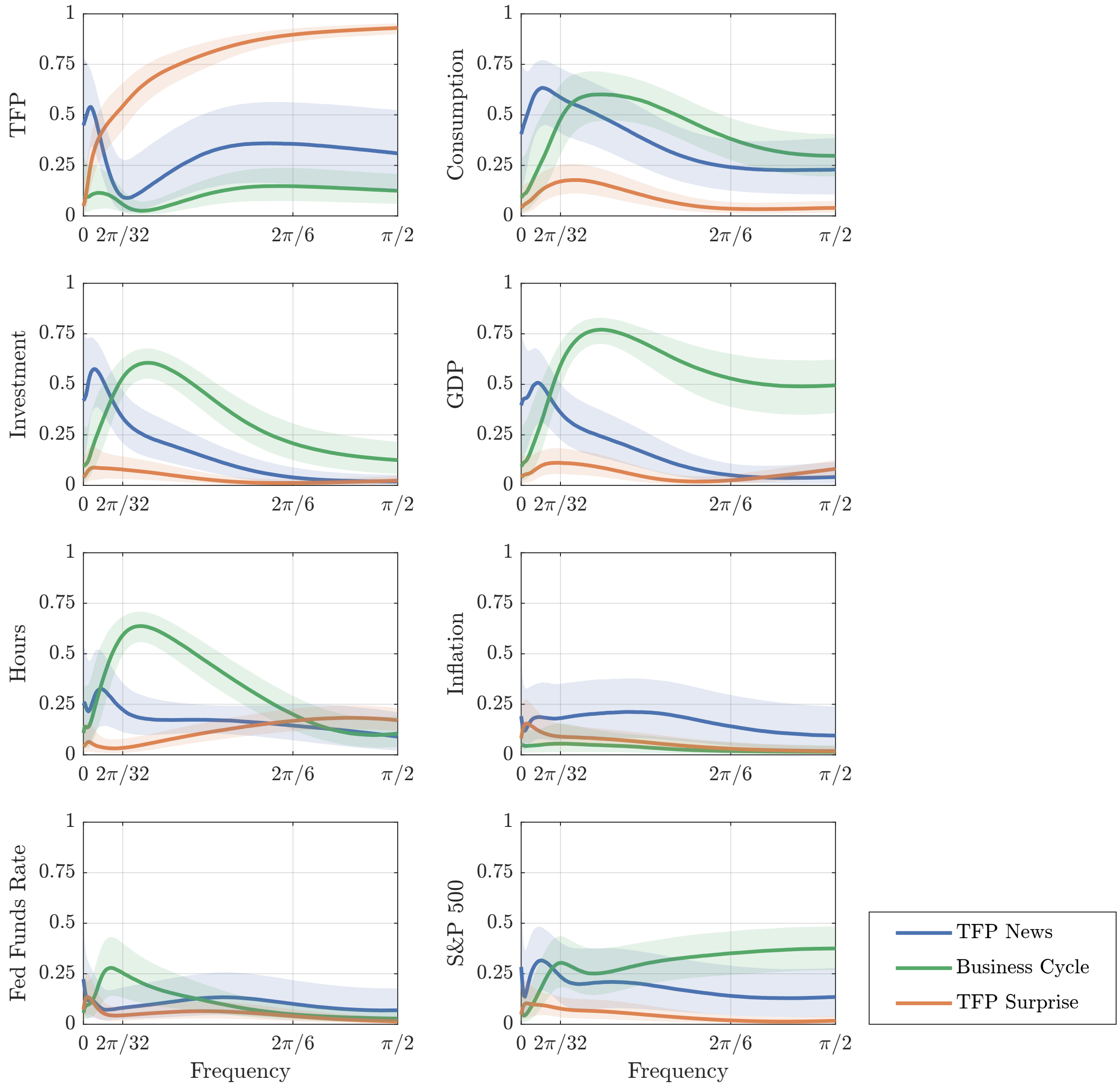

Figure B.3: Posterior estimates for FEV contributions of identified shocks in the frequency domain. **Solid lines:** Median response; **Shaded regions:** 68% error bands.

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